Square inequalities;Sign square trinomials

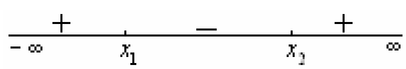
Square inequalities are forms:

$$ax^2+bx+c>0$$
$$ax^2+bx+c\geq 0$$
$$ax^2+bx+c<0$$
$$ax^2+bx+c\leq 0$$

where x is real variable (unknown) and a, b, c are real numbers, $a\neq 0$.

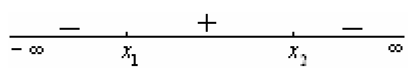
In the square function, we analyze how can look graph of square functions $y=ax^2+bx+c$,depending on the character a and D, $D=b^2-4ac$

Remind yourself:

1) $a>0, D>0\Rightarrow$ 

2) $a>0, D=0\Rightarrow y\geq 0$ always

3) $a>0, D<0\Rightarrow y>0$ always

4) $a<0, D>0\Rightarrow$ 

5) $a<0, D=0\Rightarrow y\leq 0$ always

6) $a<0, D<0\Rightarrow y<0$ always

Example 1. Determine sign of trinomials:

a) $3x^2-11x-4$
b) $-5x^2-x+4$
c) $9x^2+12x+4$
d) $-x^2-6x-9$

Solution:

a) $3x^2-11x-4$

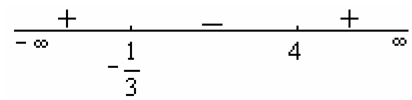
First, decide the appropriate square equality: $3x^2-11x-4=0$

$a = 3$ $\quad D = b^2 - 4ac$ $\quad x_{1,2} = \dfrac{-b \pm \sqrt{D}}{2a} = \dfrac{11 \pm 13}{6}$

$b = -11$ $\quad D = 121 + 48$ $\quad x_1 = 4$

$c = -4$ $\quad D = 169$ $\quad x_2 = -\dfrac{2}{6} = -\dfrac{1}{3}$

Since $a = 3 > 0$ **and** $D = 169 > 0$ **(first situation):**



$$3x^2 - 11x - 4 > 0 \text{ for } x \in \left(-\infty, -\frac{1}{3}\right) \cup (4, \infty)$$

$$3x^2 - 11x - 4 < 0 \text{ for } x \in \left(-\frac{1}{3}, 4\right)$$

b) $-5x^2 - x + 4$

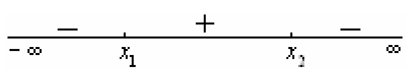
$-5x^2 - x + 4 = 0$ **See: no** multiplication and sharing with some number!

$a = -5$ $\quad D = 1 + 80$ $\quad x_{1,2} = \dfrac{1 \pm 9}{-10}$

$b = -1$ $\quad D = 81$ $\quad x_1 = -1$

$c = 4$ $\quad x_2 = \dfrac{-8}{-10} = \dfrac{4}{5}$

situation 4: $a < 0, D > 0$ so:



$$-5x^2 - x + 4 > 0 \text{ za } x \in \left(-1, \frac{4}{5}\right)$$

$$-5x^2 - x + 4 < 0 \text{ za } x \in (-\infty, -1) \cup \left(\frac{4}{5}, \infty\right)$$

c) $9x^2 + 12x + 4$

$a = 9$ $\quad D = 144 - 144$ $\quad x_{1,2} = \dfrac{-12 \pm 0}{18}$

$b = 12$ $\quad D = 0$ $\quad x_1 = -\dfrac{12}{18} = -\dfrac{2}{3}$

$c = 4$ $\quad x_2 = -\dfrac{2}{3}$

$a > 0$ and $D = 0 \rightarrow$ (situation 2) $9x^2 + 12x + 4 \geq 0$ always, $\quad (3x + 2)^2 \geq 0$

d) $-x^2-6x-9$

$$a=-1 \quad b=-6 \quad c=-9$$

$$D=36-36 \quad D=0$$

$$x_{1,2}=\frac{6\pm 0}{-2} \quad x_1=-3 \quad x_2=-3$$

$a<0$ **and** $D=0 \rightarrow -x^2-6x-9\le 0$ always, $\forall x\in R$

This we can see olso from transformation:

$$-x^2-6x-9=-(x^2+6x+9)=-(x+3)^2\le 0$$

Example 2. Solve the inequalities: $(x^2-4x-5)\cdot(x^2+2x-3)<0$

Solution:

This is a complex form of inequalities, where we can use the template already known:

$$A\cdot B<0 \Leftrightarrow (A>0, B<0)\vee(A<0, B>0)$$

Our recommendation is that such duties deal with the plates!

First, we both square equations apart on the factors, with $ax^2+bx+c=a(x-x_1)(x-x_2)$

$$x^2-4x-5=0 \Rightarrow \quad x_1=-1, \; x_2=5 \qquad x^2-4x-5=(x+1)(x-5)$$

$$x^2+2x-3=0 \Rightarrow \quad x_1=1 \; x_2=-3 \qquad x^2+2x-3=(x-1)(x+3)$$

Now look at inequalities:

$$(x+1)(x-5)(x-1)(x+3)<0$$

Make table:

	-∞				∞
x+1					
x-5					
x-1					
x+3					
(x+1)(x-5) (x-1)(x+3)					

So, each of the terms provided in the table, and the last type is"what"we should found.

Top of the line from - ∞ to ∞ we share to 5 intervals.

Above the vertical line we will enter the numbers. Which?

These numbers are solutions ofsquare equations, $x+1=0; x-5=0; x-1=0; x+3=0$
-1,5,1 and –3 ⟶ of the smallest to the largest: -3, -1,1,5

	-∞	-3	-1	1	5 ∞
x+1	-				
x-5	-				
x-1	-				
x+3	-				
(x+1)(x-5) (x-1)(x+3)					

We choose any number from each of 5 intervals and replace it in the terms x+1, x-5, x-1 and x+3; do not interest us that the number we got, but only the sign + or – who we entered in table.
For example,in the interval (- ∞, -3) select number -10, and change in:

x+1=-10-5=-9 → take - (included in the table)
x-5=-10-5= -15 → – enroll in the table
x-1=-10-11=-11 → + enroll in the table
x+3=-10+3=-7 → - enroll in the table

Between -3 and –1, select -2, etc. ... We have:

	-∞	-3	-1	1	5 ∞
x+1	-	-	+	+	+
x-5	-	-	-	-	+
x-1	-	-	-	+	+
x+3	-	+	+	+	+
(x+1)(x-5) (x-1)(x+3)	+	-	+	-	+

In this way we resolve the two inequalities:

$$(x^2-4x-5)(x^2+2x-3)<0$$
$$(x^2-4x-5)(x^2+2x-3)>0$$

As is our task to solve first, $(x^2-4x-5)(x^2+2x-3)<0$,we elect solution where the minus are:

$$\boxed{x \in (-3,-1) \cup (1,5)}$$

Example 3. Solve the inequalities: $\dfrac{x^2-3x+4}{1-x^2}>0$

Solution:

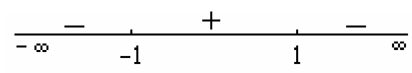
$$x^2-3x+4=0$$

$$\begin{array}{ll} a=1 & D=b^2-4ac \\ b=-3 & D=9-16 \\ c=4 & D=-7 \end{array}$$

Watch out: since $a>0$ and $D<0$ then $x^2-3x+4>0$ for $\forall x$

Therefore, must be $1-x^2>0$

$$1-x^2=0 \quad \begin{array}{l} a=-1 \\ b=0 \\ c=1 \end{array} \quad \begin{array}{l} D=0^2-4\cdot(-1)\cdot 1 \\ D=4 \end{array} \quad \begin{array}{l} x_{1,2}=\dfrac{0\pm 2}{-2} \\ x_1=-1 \\ x_2=1 \end{array}$$



Conclude: $x\in(-1,1)$ is solution!

Example 4: Find the value for x, that is $\dfrac{-x^2+2x-5}{2x^2-x-1}<-1$

Solution:

$$\frac{-x^2+2x-5}{2x^2-x-1}+1<0$$

$$\frac{-x^2+2x-5+2x^2-x-1}{2x^2-x-1}<0$$

$$\frac{x^2+x-6}{2x^2-x-1}<0$$

$$x^2+x-6=0 \Rightarrow \begin{array}{l} x_1=2 \\ x_2=-3 \end{array} \quad \Rightarrow x^2+x-6=(x-2)(x+3)$$

$$2x^2-x-1=0 \Rightarrow \begin{array}{l} x_1=1 \\ x_2=-\dfrac{1}{2} \end{array} \quad \Rightarrow 2x^2-x-1=2(x-1)\left(x+\frac{1}{2}\right)$$

Now, we have: $$\frac{(x-2)(x+3)}{2(x-1)\left(x+\frac{1}{2}\right)}<0$$

$-\infty$ $\quad -3 \quad$ $\frac{1}{2}$ $\quad 1 \quad$ 2 $\quad \infty$

x-2	-	-	-	-	+
x+3	-	+	+	+	+
x-1	-	-	-	+	+
$x+\frac{1}{2}$	-	-	+	+	+
$\frac{(x-2)(x+3)}{2(x-1)\left(x+\frac{1}{2}\right)}<0$	+	-	+	-	+

Solution: $x \in \left(-3, \frac{1}{2}\right) \cup (1,2)$

Example 5: We have $y=(r^2-1)x^2+2(r-1)x+2$. Determine the real parameter r, function to be positive for any real x.

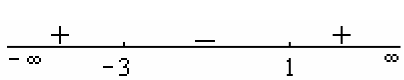
Solution:

$(r^2-1)x^2+2(r-1)x+2>0$

function is positive if: $a>0$ and $D<0$

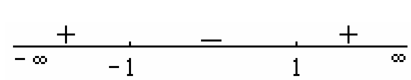
So: $a=r^2-1$
$b=2(r-1)$
$c=2$

$-4r^2-8r+12<0 / :(-4)$
$r^2+2r-3>0$
$r_{1,2}=\frac{-2\pm 4}{2}$
$r_1=1$
$r_2=-3$



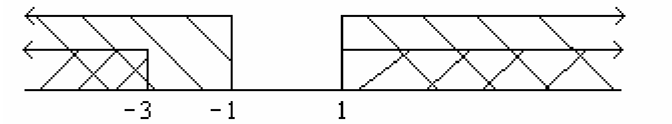
$r \in (-\infty,-3) \cup (1,\infty)$

$a>0$
$r^2-1>0$
$r^2-1=0$
$r_1=-1$
$r_2=1$



$r \in (-\infty,-1) \cup (1,\infty)$

Final solution:



$r \in (-\infty, -3) \cup (1, \infty)$

Example 6. Solve the inequalities $rx^2 + 2(r+2)x + 2r + 4 < 0$ **for all real value r.**

Solution:

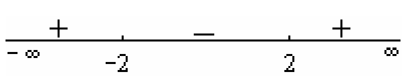
$rx^2 + 2(r+2)x + 2r + 4 < 0$

Must be: $a < 0$ and $D < 0$

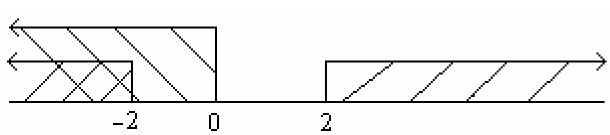
$$a = r$$
$$b = 2(r+2)$$
$$c = 2r + 4$$

$$D = b^2 - 4ac$$
$$D = [2(r+2)]^2 - 4 \cdot r(2r+4)$$
$$D = 4(r+2)^2 - 4r(2r+4)$$
$$D = 4(r^2 + 4r + 4) - 8r^2 - 16r$$
$$D = 4r^2 + 16r + 16 - 8r^2 - 16r$$
$$D = -4r^2 + 16$$

$$-4r^2 + 16 < 0 / : (-4)$$
$$r^2 - 4 > 0$$
$$r_1 = 2$$
$$r_2 = -2$$



$r \in (-\infty, -2) \cup (2, \infty)$ and $a < 0$, $r < 0$



Final solution: $r \in (-\infty, -2)$

Example 7. **Solve:** $\left|\frac{x^2+kx+1}{x^2+x+1}\right|<2$

Solution:

$$\left|\frac{x^2+kx+1}{x^2+x+1}\right|<2$$

$$\left|\frac{x^2+kx+1}{x^2+x+1}\right|<2 \Rightarrow -2<\frac{x^2+kx+1}{x^2+x+1}<2$$

Therefore, this task requires solving two inequalities:

1)

$$-2<\frac{x^2+kx+1}{x^2+x+1}$$

$$\frac{x^2+kx+1}{x^2+x+1}+2>0$$

$$\frac{x^2+kx+1+2x^2+2x+2}{x^2+x+1}>0$$

$$\frac{3x^2+x(k+2)+3}{x^2+x+1}>0$$

$$x^2+x+1=0$$

$a=1$ $\quad D=b^2-4ac$

$b=1$ $\quad D=1-4$

$c=1$ $\quad D=-3$

$a>0$ and $D<0 \Rightarrow x^2+x+1>0$ for $\forall x$

$3x^2+x(k+2)+3=0$, $\quad 3x^2+x(k+2)+3>0$ for $a>0, D<0$

$a=3$ $\quad D=(k+2)^2-4\cdot 3\cdot 3$

$b=k+2$ $\quad D=k^2+4k+4-36$

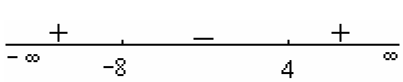
$c=3$ $\quad D=k^2+4k-32$

$$k^2+4k-32<0$$

$$k^2+4k-32=0$$

$$k_1=4$$

$$k_2=-8$$



$k\in(-8,4)$

2)

$$\frac{x^2+kx+1}{x^2+x+1}<2 \Rightarrow \frac{x^2+kx+1}{x^2+x+1}-2<0$$

$$\frac{x^2+kx+1-2x^2-2x-2}{x^2+x+1}<0$$

$$\frac{-x^2+(k-2)x-1}{x^2+x+1}<0$$

$x^2+x+1>0$,So:

$$-x^2+(k-2)x-1<0/(-1)$$

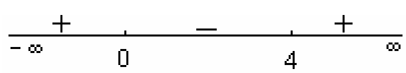
$$x^2-(k-2)x+1>0$$

$$D<0 \Rightarrow \quad D=\left[-(k-2)\right]^2-4$$

$$D=k^2-4k+4-4$$

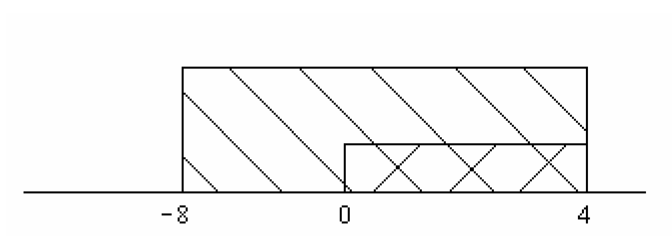
$$D=k^2-4k<0$$

$$k^2-4k=0 \Rightarrow k_1=0, k_2=4$$



$k \in (0,4)$

Those two solutions $k \in (-8,4)$ and $k \in (0,4)$ give as final solution:



$$\boxed{k \in (0,4)}$$